

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 10

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Homework 10.1 (Riemann sphere). The goal of this exercise is to introduce calculus “at infinity”. We set $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where for the moment ∞ is an abstract element. We say a sequence $(z_n)_{n \in \mathbb{N}}$ in $\widehat{\mathbb{C}}$ converges to a point $z \in \widehat{\mathbb{C}}$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

- $z_n = \infty$ or $z_n \in \mathbb{C}$ yet $|z_n| \geq 1/\varepsilon$ provided $z = \infty$ and
- $z_n \in \mathbb{C}$ and $|z_n - z| \leq \varepsilon$ provided $z \in \mathbb{C}$.

Moreover, let $\mathbf{S}^2 := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the usual two-dimensional unit sphere and define the stereographic projection $P: \mathbf{S}^2 \rightarrow \widehat{\mathbb{C}}$ through

$$P(x) := \begin{cases} \frac{x_1}{1-x_3} + i \frac{x_2}{1-x_3} & \text{if } x_3 \neq 1, \\ \infty & \text{otherwise.} \end{cases}$$

- a. Show P is a homeomorphism (where continuity is tacitly understood as sequential continuity).
- b. Conclude $\widehat{\mathbb{C}}$ is sequentially compact.

Homework 10.2 (Open mapping theorem for the Riemann sphere). Let $\widehat{D} \subset \widehat{\mathbb{C}}$ be a domain and let $f: \widehat{D} \rightarrow \widehat{\mathbb{C}}$ be holomorphic and nonconstant. Show $f(\widehat{D})$ is again a domain.

Homework 10.3 (Extension of entire functions*). In this exercise we show polynomials are the only entire functions that can be extended to the Riemann sphere in a holomorphic way.

- a. Let $P: \mathbb{C} \rightarrow \mathbb{C}$ be a nonconstant polynomial. Show that setting $P(\infty) := \infty$ defines a holomorphic extension $P: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$.
- b. Show if $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is holomorphic and satisfies $f(\mathbb{C}) \subset \mathbb{C}$ then f is a polynomial¹.

Homework 10.4 (Holomorphic functions on the Riemann sphere are rational). Let the function $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be holomorphic. Show f is a rational function², i.e. there are polynomials $P, Q: \mathbb{C} \rightarrow \mathbb{C}$ such that for every $z \in \mathbb{C} \setminus \{f = \infty\}$,

$$f(z) = \frac{P(z)}{Q(z)}.$$

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¹**Hint.** Consider the assignment $z \mapsto f(1/z)$ and its singularity at 0.

²**Hint.** You may need the following generalized Liouville theorem. If $g: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and there exist $R > 0$ and $n \in \mathbb{N}$ such that $|g(z)| \leq R|z|^n$ for every $z \in \bar{B}_R(0)$, then g is a polynomial no larger than n .